PROBLEM: Show that the numerical values of the following expressions in indicial notation are as given: **(a)** $\delta_{ij}\delta_{ij} = 3$, **(b)** $\delta_{ij}e_{ijk} = 0$, **(c)** $e_{ijk}e_{ijk} = 6$ and **(d)** $e_{ijk}\sigma_{jk} = 0$. The ranges on all indices are 1, 2, 3. Use the summation convention and the definition of the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

and the permutation symbol

$$e_{ijk} = \begin{cases} 1 & \text{for positive permutations} \\ 0 & \text{if any two indices are equal} \\ -1 & \text{for negative permutations} \end{cases}$$

where *ijk*, *jki* and *kij* are positive and *ikj*, *kji* and *jik* are negative permutations of indices.

SOLUTION

(a) The indices *i* and *j* are repeated; sum on *i* first, then on *j*.

$$\begin{aligned} \delta_{ij}\delta_{ij} &= \delta_{1j}\delta_{1j} + \delta_{2j}\delta_{2j} + \delta_{3j}\delta_{3j} \\ &= \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13} + \delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23} + \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33} \\ &= (1)(1)+(0)(0)+(0)(0)+(0)(0)+(1)(1)+(0)(0)+(0)(0)+(0)(0)+(1)(1) \\ &= 3 \end{aligned}$$

(b) The indices *i* and *j* are repeated; sum on *i* first, then on *j*.

$$\begin{aligned} \delta_{ij}e_{ijk} &= \delta_{1j}e_{1jk} + \delta_{2j}e_{2jk} + \delta_{3j}e_{3jk} \\ &= \delta_{11}\cancelto{0}{e_{11k}} + \cancelto{0}{\delta_{12}}e_{12k} + \cancelto{0}{\delta_{13}}e_{13k} + \cancelto{0}{\delta_{21}}e_{21k} + \delta_{22}\cancelto{0}{e_{22k}} + \cancelto{0}{\delta_{23}}e_{23k} + \cancelto{0}{\delta_{31}}e_{31k} + \cancelto{0}{\delta_{32}}e_{32k} + \delta_{33}\cancelto{0}{e_{33k}} \\ &= 0 \end{aligned}$$

(c) The indices *i*, *j* and *k* are repeated; sum on *i* first, then on *j* and finally on *k*.

$$\begin{aligned} e_{ijk}e_{ijk} &= e_{1jk}e_{1jk} + e_{2jk}e_{2jk} + e_{3jk}e_{3jk} \\ &= \cancelto{0}{e_{11k}}\cancelto{0}{e_{11k}} + e_{12k}e_{12k} + e_{13k}e_{13k} + e_{21k}e_{21k} + \cancelto{0}{e_{22k}}\cancelto{0}{e_{22k}} + e_{23k}e_{23k} + e_{31k}e_{31k} + e_{32k}e_{32k} + \cancelto{0}{e_{33k}}\cancelto{0}{e_{33k}} \\ &= \cancelto{0}{e_{121}}\cancelto{0}{e_{121}} + \cancelto{0}{e_{122}}\cancelto{0}{e_{122}} + e_{123}e_{123} + \cancelto{0}{e_{131}}\cancelto{0}{e_{131}} + e_{132}e_{132} + \cancelto{0}{e_{133}}\cancelto{0}{e_{133}} + \cancelto{0}{e_{211}}\cancelto{0}{e_{211}} + \cancelto{0}{e_{212}}\cancelto{0}{e_{212}} + e_{213}e_{213} \\ &\quad + e_{231}e_{231} + \cancelto{0}{e_{232}}\cancelto{0}{e_{232}} + \cancelto{0}{e_{233}}\cancelto{0}{e_{233}} + \cancelto{0}{e_{311}}\cancelto{0}{e_{311}} + e_{312}e_{312} + \cancelto{0}{e_{313}}\cancelto{0}{e_{313}} + e_{321}e_{321} + \cancelto{0}{e_{322}}\cancelto{0}{e_{322}} + \cancelto{0}{e_{323}}\cancelto{0}{e_{323}} \\ &= (1)(1)+(-1)(-1)+(-1)(-1)+(1)(1)+(1)(1)+(-1)(-1) \\ &= 6 \end{aligned}$$

(d) The indices *j* and *k* are repeated; sum on *j* first and then on *k*.

$$
\begin{aligned}
e_{ijk}\sigma_{jk} &= e_{i1k}\sigma_{1k} + e_{i2k}\sigma_{2k} + e_{i3k}\sigma_{3k} \\
&= \cancelto{0}{e_{i11}}\sigma_{11} + e_{i12}\sigma_{12} + e_{i13}\sigma_{13} + e_{i21}\sigma_{21} + \cancelto{0}{e_{i22}}\sigma_{22} + e_{i23}\sigma_{23} + e_{i31}\sigma_{31} + e_{i32}\sigma_{32} + \cancelto{0}{e_{i33}}\sigma_{33} \\
&= e_{i12}\sigma_{12} + e_{i13}\sigma_{13} + e_{i21}\sigma_{21} + e_{i23}\sigma_{23} + e_{i31}\sigma_{31} + e_{i32}\sigma_{32}
\end{aligned}
$$

Since σ_{jk} is symmetric such that $\sigma_{jk} = \sigma_{kj}$

$$e_{ijk}\sigma_{jk} = \left(e_{i12} + e_{i21}\right)\sigma_{12} + \left(e_{i13} + e_{i31}\right)\sigma_{13} + \left(e_{i23} + e_{i32}\right)\sigma_{23}$$

When $i = 1$

$$e_{1jk}\sigma_{jk} = \left(\cancelto{0}{e_{112}} + \cancelto{0}{e_{121}}\right)\sigma_{12} + \left(\cancelto{0}{e_{113}} + \cancelto{0}{e_{131}}\right)\sigma_{13} + \left(\cancelto{1}{e_{123}} + \cancelto{-1}{e_{132}}\right)\sigma_{23} = 0$$

When $i = 2$

$$e_{2jk}\sigma_{jk} = \left(\cancelto{0}{e_{212}} + \cancelto{0}{e_{221}}\right)\sigma_{12} + \left(\cancelto{-1}{e_{213}} + \cancelto{1}{e_{231}}\right)\sigma_{13} + \left(\cancelto{0}{e_{223}} + \cancelto{0}{e_{232}}\right)\sigma_{23} = 0$$

When $i = 3$

$$e_{3jk}\sigma_{jk} = \left(\cancelto{1}{e_{312}} + \cancelto{-1}{e_{321}}\right)\sigma_{12} + \left(\cancelto{0}{e_{313}} + \cancelto{0}{e_{331}}\right)\sigma_{13} + \left(\cancelto{0}{e_{323}} + \cancelto{0}{e_{332}}\right)\sigma_{23} = 0$$

Therefore, $e_{ijk}\sigma_{jk} = 0$.