PROBLEM: A one dimensional prismatic bar of mass m has an undeformed length L_0, cross sectional area A_0 and density ρ_0. It carries a Lagrange stress $T_{11} = F/A_0$ when subjected to an axial force F. Show that the Cauchy and Kirchoff stresses are, respectively, given by

$$\sigma_{11} = \frac{F}{A} \qquad\qquad S_{11} = \frac{L_0}{L}\frac{F}{A_0}$$

SOLUTION: The deformation and inverse mappings for this problem have previously been found to be

$$x = \frac{L}{L_0}a \qquad\qquad a = \frac{L_0}{L}x$$

The mass remains constant throughout the deformation, therefore the original and deformed densities are

$$\rho_0 = \frac{m}{A_0 L_0} \qquad\qquad \rho = \frac{m}{AL}$$

The (existing) Cauchy stress in terms of the (existing) Lagrange stress is given by

$$\sigma_{ji} = \frac{\rho}{\rho_0}\frac{\partial \hat{x}_i}{\partial a_k}T_{kj} \quad \Rightarrow \quad \sigma_{11} = \frac{\rho}{\rho_0}\frac{\partial \hat{x}}{\partial a}T_{11} = \frac{\dfrac{m}{AL}}{\dfrac{m}{A_0 L_0}}\left[\frac{\partial}{\partial a}\left(\frac{L}{L_0}a\right)\right]\frac{F}{A_0} \quad \Rightarrow \quad \boxed{\sigma_{11} = \frac{F}{A}}$$

The (existing) Kirchoff stress in terms of the (existing) Lagrange stress is given by

$$S_{ji} = \frac{\rho}{\rho_0}\frac{\partial \hat{a}_i}{\partial x_k}T_{jk} \quad \Rightarrow \quad S_{11} = \frac{\partial \hat{a}}{\partial x}T_{11} = \left[\frac{\partial}{\partial a}\left(\frac{L_0}{L}x\right)\right]\frac{F}{A_0} \quad \Rightarrow \quad \boxed{S_{11} = \frac{L_0}{L}\frac{F}{A_0}}$$