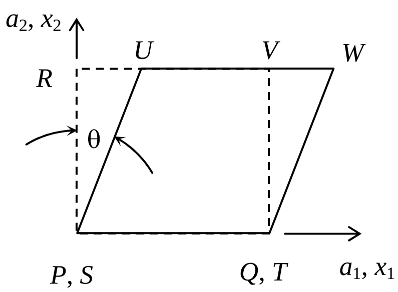
PROBLEM: Consider the simple shear of the originally unit square plate shown in the undeformed (left) and deformed (right) configurations. **(a)** Determine expressions for the Green, Almansi and Cauchy strain components in terms of θ. **(b)** Prepare a table of the numerical values of the nonzero strain components for $\theta = 30^\circ$, 3°, 0.3°, and 0.03°. **(c)** Plot the nonzero strain components as a function of θ. **(d)** If this plate were made of annealed 316L stainless steel, for what value of θ would you consider this deformation to be infinitesimal and why?

SOLUTION: The corners of the plate in the undeformed (*PRVQ*) and deformed (*SUWT*) configurations are labeled as shown. The coordinates and complementary corners are given by



$$
\begin{aligned}
P &: (0,0) \to S : (0,0) \\
Q &: (1,0) \to T : (1,0) \\
R &: (0,1) \to U : (\tan\theta, 1) \\
V &: (1,1) \to W : (1+\tan\theta, 1)
\end{aligned}
$$

(a) The deformation mapping is found from these coordinates and the generalized homogeneous mappings

$$x_1 = \hat{x}_1(a_i) = c_1 a_1 + c_2 a_2 + c_3 \qquad x_2 = \hat{x}_2(a_i) = c_4 a_1 + c_5 a_2 + c_6 \qquad x_3 = \hat{x}_3(a_i) = a_3$$

and by noting that the problem is planar. Substituting the coordinates and solving the resulting system of equations yields the <u>deformation mapping</u>

$$x_1 = a_1 + a_2 \tan\theta \qquad x_2 = a_2 \qquad x_3 = a_3$$

such that the <u>inverse mapping</u> is

$$a_1 = \hat{a}_1(x_i) = x_1 - x_2 \tan\theta \qquad a_2 = \hat{a}_2(x_i) = x_2 \qquad a_3 = \hat{a}_3(x_i) = x_3$$

The <u>Green strains</u> are given by

$$E_{ij} = \frac{1}{2}\left(\delta_{mn}\frac{\partial \hat{x}_m}{\partial a_i}\frac{\partial \hat{x}_n}{\partial a_j} - \delta_{ij}\right) = \frac{1}{2}\left(\frac{\partial \hat{x}_1}{\partial a_i}\frac{\partial \hat{x}_1}{\partial a_j} + \frac{\partial \hat{x}_2}{\partial a_i}\frac{\partial \hat{x}_2}{\partial a_j} + \frac{\partial \hat{x}_3}{\partial a_i}\frac{\partial \hat{x}_3}{\partial a_j} - \delta_{ij}\right)$$

and the only nonzero components are

$$\boxed{E_{12} = \frac{1}{2}\tan\theta} \qquad \boxed{E_{22} = \frac{1}{2}\tan^2\theta}$$

The Almansi strains are given by

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - \delta_{mn}\frac{\partial \hat{a}_m}{\partial x_i}\frac{\partial \hat{a}_n}{\partial x_j}\right) = \frac{1}{2}\left(\delta_{ij} - \frac{\partial \hat{a}_1}{\partial x_i}\frac{\partial \hat{a}_1}{\partial x_j} - \frac{\partial \hat{a}_2}{\partial x_i}\frac{\partial \hat{a}_2}{\partial x_j} - \frac{\partial \hat{a}_3}{\partial x_i}\frac{\partial \hat{a}_3}{\partial x_j}\right)$$

and the only nonzero components are

$$\boxed{e_{12} = \frac{1}{2}\tan\theta} \qquad \boxed{e_{22} = -\frac{1}{2}\tan^2\theta}$$

The Cauchy infinitesimal strains are given by

$$\varepsilon_{ij} = \delta_{ij} - \frac{1}{2}\left(\frac{\partial \hat{a}_j}{\partial x_i} + \frac{\partial \hat{a}_i}{\partial x_j}\right)$$

and the only nonzero component is

$$\boxed{\varepsilon_{12} = \frac{1}{2}\tan\theta}$$

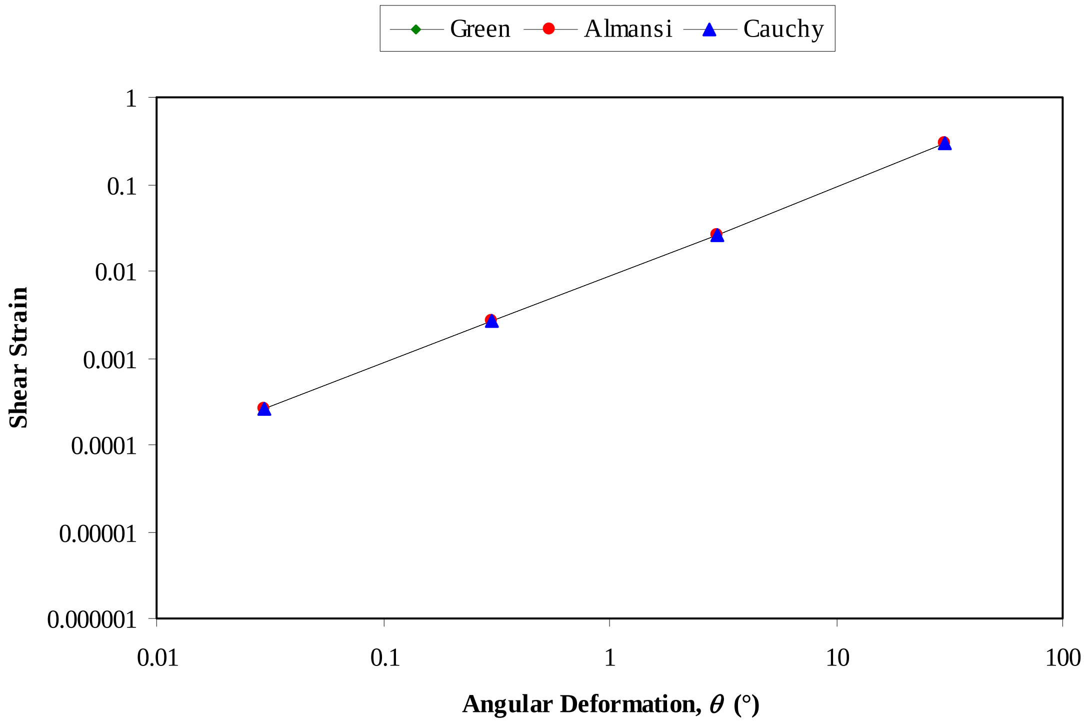
(b) The nonzero shear strain components are tabulated below.

Shear Strain	**Angular Deformation, θ**			
	0.03°	**0.3°**	**3°**	**30°**
Green E_{12}	0.000 262	0.002 618	0.026 20	0.288 7
Almansi e_{12}	0.000 262	0.002 618	0.026 20	0.288 7
Cauchy ε_{12}	0.000 262	0.002 618	0.026 20	0.288 7

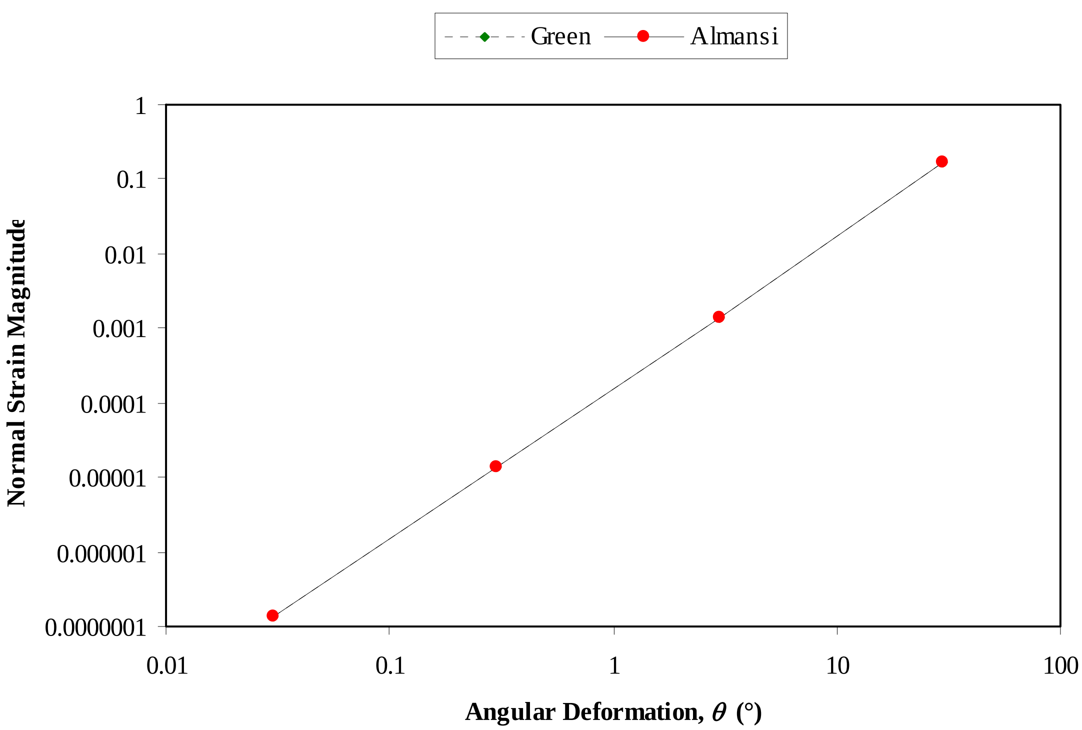
and the nonzero normal strain components are tabulated below.

Normal Strain	**Angular Deformation, θ**			
	0.03°	**0.3°**	**3°**	**30°**
Green E_{22}	0.000 000 137	0.000 013 7	0.001 37	0.167
Almansi e_{22}	-0.000 000 137	-0.000 013 7	-0.001 37	-0.167

(c) A plot of the shear strains versus angular deformation is given below. All strain measures return the same value for each value of θ.



A plot of the normal strain magnitudes (because the Almansi strains are negative) versus angular deformation is given below. The Green and Almansi strain measures return the same magnitude for each value of θ.



(d) Annealed 316L stainless steel has a shear yield strain $\gamma_Y \approx 2,000\ \mu\varepsilon$. The engineering shear strain is twice the Cauchy shear strain and should be less than γ_Y for the shear strain to be considered infinitesimal. Thus, the maximum angular deformation considered to be infinitesimal is found from

$$2\varepsilon_{12} = \tan\theta \le \gamma_Y \approx 2,000\ \mu\varepsilon \quad \Rightarrow \quad \boxed{\theta \le 0.115^\circ}$$

This value for θ gives the following normal strain components

$$E_{22} = 2\ \mu\varepsilon \qquad\qquad e_{22} = -2\ \mu\varepsilon$$

which are very close to zero (as $\varepsilon_{22} = 0$). So, it seems reasonable that this is an acceptable upper limit on the angular deformation.