MER 440 ORTHOPEDIC BIOMECHANICS

CRACK GROWTH AND FATIGUE

Paris Crack Growth Law

These notes address the analysis of crack growth under fatigue loading conditions. Paul C. Paris in 1964 and before was the first to reduce crack growth data by plotting (on a log-log scale) the crack growth rate versus the stress intensity factor range. The crack growth rate is

$$\frac{da}{dn}$$

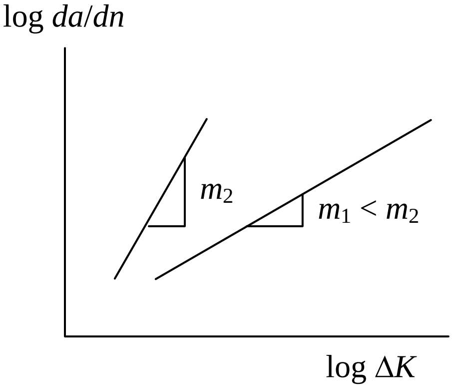
where a is the crack length after n number of cycles of loading. The stress intensity factor range is given by

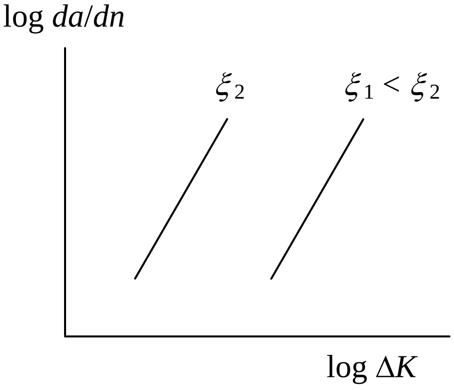
$$\Delta K = K_{\max} - K_{\min} = \beta\sqrt{\pi a}\,\Delta\sigma$$

where $K_{\max}$ and $K_{\min}$ are the maximum and minimum stress intensity factors corresponding to the maximum and minimum stress levels $\sigma_{\max}$ and $\sigma_{\min}$, and the expression on the right hand side is the general form (where β accounts for geometry) for the stress intensity factor (using the stress range $\Delta\sigma$). Paris found that a plot of experimental data on a log-log scale generally followed a straight line given by

$$\log\frac{da}{dn} = m\log\Delta K + \log\xi$$

where the parameters m (the slope) and ξ are chosen (from least squares regression) to best fit the data. Hypothetical plots are given below to depict the effect of varying these parameters.





Taking the antilog of each side yields the Paris crack growth (power) law given by

$$\boxed{\frac{da}{dn} = \xi\left(\Delta K\right)^m}$$

The Paris Law fits data well for intermediate ΔK's but not when ΔK is low (there appears to be a threshold ΔK_{th} below which no propagation occurs, akin to an endurance limit) or when ΔK is high (where cracks propagate rapidly). It also does not explicitly account for the effect of varying the stress ratio R, to reflect that with more tension in the cyclic loading, there will be more rapid crack growth. Finally, the effect of time spent in compression is ignored; therefore

$$\Delta\sigma = \sigma_{\max} \quad \text{if} \quad \sigma_{\max} < 0$$

Practical Implementation

It is assumed that the parameters m and ξ have been determined for a particular material. It is now desired to determine the number of cycles n it will take to grow a crack of an original specified length a_i to a final specified length a_f. The original length might be the minimum crack length detectable by nondestructive inspection, and the final length might be the critical crack length. The variables in the Paris Law are separated

$$dn = \frac{da}{\xi\left(\Delta K\right)^m} = \frac{da}{\xi\left(\beta\sqrt{\pi a}\Delta\sigma\right)^m}$$

and integrated between a_i and a_f. A general form for this integration can not be written because, in general, β is a function of the crack length a as well.

Example

An estimate for m = 7.3219 and ξ = 0.00001089 (units will be addressed later) can be made using an interpretation from Taylor (*J Biomech* 1998) of data from Schaffler et al. (*J Biomech* 1989). From the critical half-penny crack size example at $\theta = 90°$,

$$\Delta K_I = 2.422\Delta\sigma_o\sqrt{\frac{a}{\pi}}$$

Substituting into the crack growth law and separating variables yields

$$dn = \frac{da}{\xi\left(2.422\sqrt{\frac{a}{\pi}}\sigma_o\right)^m}$$

Integrating yields

$$n = \frac{1}{\xi}\left(\frac{2.422}{\sqrt{\pi}}\Delta\sigma_o\right)^{-m}\left(\frac{2}{2-m}\right)a^{\frac{2-m}{2}}\Bigg|_{a_i}^{a_f}$$

Note that the units on the initial a_i and final a_f crack lengths must be [m] if the stress range $\Delta\sigma_o$ is in units of [MPa]. The values given previously for the crack growth law parameters m and ξ are consistent with these units. An appropriate choice for the initial crack length is the smallest detectable crack length in a structural application; for bone, an appropriate choice is the crack length that triggers remodeling, i.e., on the order of 100 μm. An appropriate choice for the final crack length is the critical crack length. Therefore, with these limits and the stress range, we can determine the number of cycle required to grow the crack from some detectable size to the size that causes catastrophic failure.

Substituting all of the numerical values, the required number of cycles for each stress range is tabulated below.

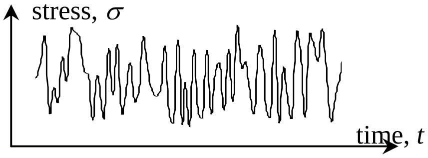
strain range $\Delta\varepsilon$	stress range $\Delta\sigma$	critical crack size a_{cr}	number of cycles n
1,000 με	20 MPa	21.4 mm	45,987 cycles*
2,000 με	40 MPa	5.36 mm	287 cycles
3,000 με	60 MPa	2.38 mm	15 cycles

The asterisk on the number of cycles for the lowest stress range case denotes that the critical crack size was set equal to the thickness of the tibial cortex (1 cm) since the critical crack size for this case is predicted to be twice this cortical thickness.

This analysis seems to indicate that we are all teetering on the edge of destruction! Fortunately, two aspects of crack growth in bone work in our favor. First, the next barrier to crack growth, a cement line, is close by. Second, the crack growth rate actually diminishes in bone as a microcrack approaches a barrier such as a cement line.

Variable Amplitude Loading

Unless you are walking on level ground with a steady gait at constant velocity ... then your tibia rarely experiences constant amplitude loading. This is true for all of the skeletal elements. The loading record or spectrum in a region of a skeletal element may appear quire random, as depicted in the figure. From one "interval" to the next, the stress and stress intensity factor range vary widely. Methods exist to accommodate such variable amplitude loading in *S*-*N* curves and crack growth laws. In some materials, e.g., ductile metals, a large load excursion (hence, large $\Delta\sigma$ and large ΔK) creates a relatively large plastic zone ahead of an advancing crack tip thereby diminishing the effect of subsequent lower loads. This, however, does not seem to be true for bone, which is relatively brittle.



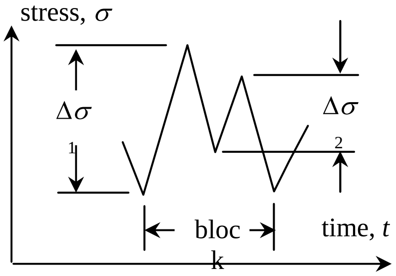
If the order of the loading intervals is not a concern (i.e., has no effect on crack growth and fatigue life), then a "characteristic ΔK" method is used to accommodate variable amplitude loading. One such method uses the root-mean (RM) of ΔK, denoted by ΔK_{RM}, as the characteristic ΔK, defined as

$$\Delta K_{\mathrm{RM}} \equiv \left[\frac{\sum (\Delta K_i)^m n_i}{\sum n_i} \right]^{1/m}$$

where *m* is the exponent from the constant amplitude crack growth law. For example, given the all tension "block" of loading depicted in the below plot, ΔK_{RM} is given by

$$\Delta K_{\mathrm{RM}} = \left[\frac{(\Delta K_1)^m (1) + (\Delta K_2)^m (1)}{1+1} \right]^{1/m}$$

$$= \left\{ \frac{1}{2} \left[(\Delta K_1)^m + (\Delta K_2)^m \right] \right\}^{1/m}$$



Note that if $m = 2$, the root-mean becomes the root-mean-square (RMS). Finally, if it just so happened to be true that $m = 1$, then the root-mean stress intensity factor range ΔK_{RM} becomes the arithmetic mean, as in

$$\Delta K_{\mathrm{RM}} \big|_{m=1} = \frac{1}{2} (\Delta K_1 + \Delta K_2)$$