TENSORS

Tensors

Tensors are mathematical objects. They possess specific components for every coordinate system, which (in general) change under coordinate transformation. Cartesian tensors are tensors referred to rectangular Cartesian coordinates in three dimensional space, in which a coordinate system is typically represented by x_1 , x_2 , x_3 (or x, y, z or 1, 2, 3) axes. Tensors are usually written using (the subscripted) index notation. The tensor order is given by the number of live (unrepeated) subscripts. If a subscript is repeated, then the summation convention applies, that is, summation occurs over the range of that subscript. For example, the Kronecker delta is an important function in mechanics and is given by

$$\delta_{ij} = \begin{cases} \delta_{12} = \delta_{13} = \delta_{23} = \delta_{21} = \delta_{31} = \delta_{32} \equiv 0 & (i \neq j) \\ \delta_{11} = \delta_{22} = \delta_{33} \equiv 1 & (i = j) \end{cases}$$

where i and j range from 1 to 3; therefore, no summation occurs over i or j because neither are repeated. A quantity called the first stress invariant is given by

$$\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

where it is noted that since i is repeated, summation occurs over it.

Some examples of tensors are:

Object	Variable	Tensor Order	Other Name	Number of Live Subscripts
Temperature	T	zero	scalar	0
First Stress Invariant	σ_{ii}	zero	scalar	0
Force	V_i	first	vector	1
Stress	σ	second	—	2
Elastic Constants	C^*_{ijkl}	fourth	—	4

Tensor Transformation

Tensor transformation is governed by the applicable transformation equation, which involves direction cosines a_{ij} (i, j = 1, 2, 3). Usually, the reference coordinate system is called the unprimed (old, original) system, and the transformed system is called the primed (new) system (Figure 1). The direction cosines are the cosine of the angles measured from the unprimed axes to the primed axes. For example, $a_{13} = \cos\theta_{13}$ and is equal to the cosine of the angle θ_{13} measured from the x_1 to the x_3' axes.

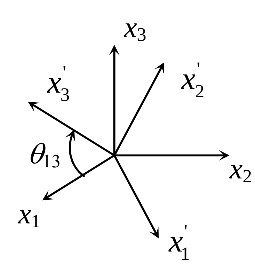

Figure 1. Rectangular Cartesian coordinate systems.

An orderly way to determine the direction cosines is to construct a matrix of direction cosines

$$a_{ij} = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{ccc|c} x_1' & x_2' & x_3' & \\ \hline a_{11} & a_{12} & a_{13} & x_1 \\ a_{21} & a_{22} & a_{23} & x_2 \\ a_{31} & a_{32} & a_{33} & x_3 \end{array}$$

For example, the transformation equation for a vector (such as a force) is

$$V_i' = a_{ji} V_j$$

Since j is repeated, the expanded form is

$$V_i' = a_{1i} V_1 + a_{2i} V_2 + a_{3i} V_3$$

or, in terms of direction cosines,

$$V_i' = \cos\theta_{1i} V_1 + \cos\theta_{2i} V_2 + \cos\theta_{3i} V_3$$

The transformation equation for a second order tensor is

$$\sigma_{ij}' = a_{ki} a_{lj} \sigma_{kl}$$

Since both k and l are repeated, a more orderly expansion is warranted, such as first expanding on k and then expanding on l. First, expanding on k yields

$$\sigma_{ij}' = a_{1i} a_{lj} \sigma_{1l} + a_{2i} a_{lj} \sigma_{2l} + a_{3i} a_{lj} \sigma_{3l}$$

where the first term on the right hand side is from k = 1, the second from k = 2, and the third from k = 3. Expanding each term on l (because l is repeated in each of the three terms on the right hand side) yields

$$\begin{aligned} \sigma_{ij}' &= a_{1i} a_{1j} \sigma_{11} + a_{1i} a_{2j} \sigma_{12} + a_{1i} a_{3j} \sigma_{13} \\ &+ a_{2i} a_{1j} \sigma_{21} + a_{2i} a_{2j} \sigma_{22} + a_{2i} a_{3j} \sigma_{23} \\ &+ a_{3i} a_{1j} \sigma_{31} + a_{3i} a_{2j} \sigma_{32} + a_{3i} a_{3j} \sigma_{33} \end{aligned}$$

The transformation equation for a fourth order tensor is

$$C_{ijkl}^{*'} = a_{mi} a_{nj} a_{rk} a_{sl} C_{mnrs}^{*}$$

In this equation, m, n, r, and s are repeated indices, and a full expansion involves 81 terms.