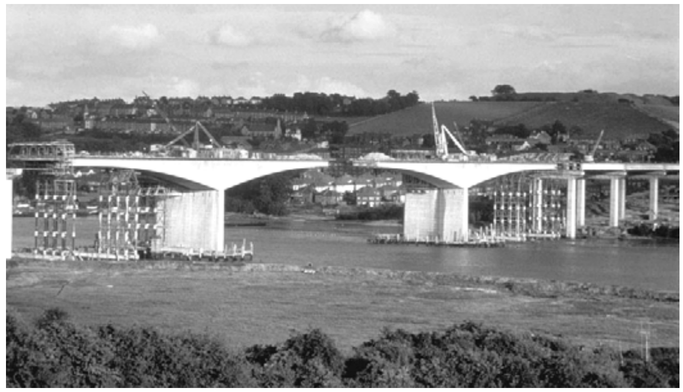
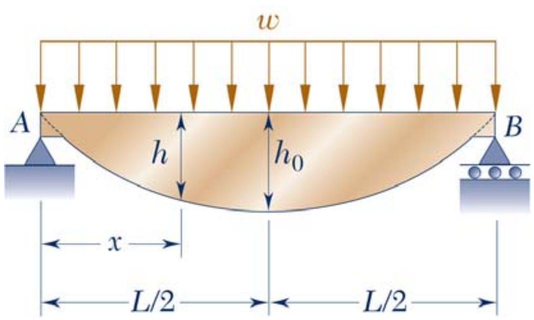
INTRODUCTION: Prismatic beams are of constant cross-section and are often over-designed along their length because critical regions where the stresses are greatest do not exist everywhere. In some cases, non-prismatic beams can be used so that the material exists only where it is needed. Such beams can achieve great weight savings in applications where this is important, e.g., spaceflight and the self-weight of bridges (figure at right).

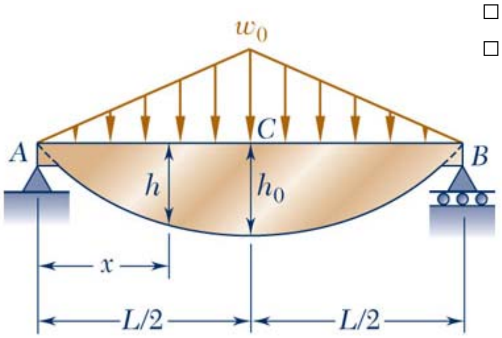


EXERCISE: Each group of two students is to design a non-prismatic beam, based on flexural stress, that carries the greatest resultant load P for the lightest beam weight W, i.e., a beam with a maximum greatest performance ratio $\Pi = P/W$. Each group is to choose one beam from the below figures and one cross section: square or circular. Your instructor will assist in the selections so that a desired "overlap" exists among the groups and lab sections. Determine the performance ratio Π_o of a baseline beam that fits within a maximum dimension envelope, i.e., a beam of constant cross section of dimension h_o = 25 mm and L = 1 m long with an allowable stress σ_{allow} = 100 MPa. Determine the increase in performance ratio realized by the design of your non-prismatic beam. Finally, address the shear stresses near the supports, i.e., the average shear stress must be multiplied by 1.7 for square and 1.4 for circular cross sections to determine the maximum shear stress.

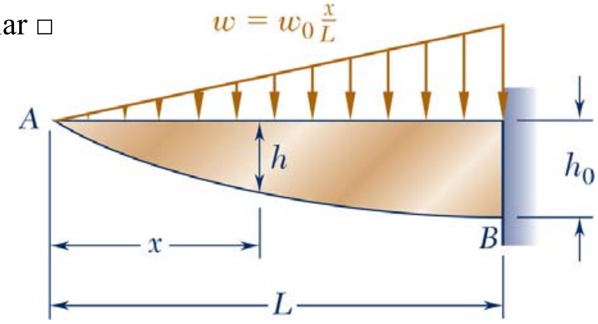
REPORTING REQUIREMENTS: Each student is to submit a report (two 8.5" x 11" pages maximum, 1" margins all around, 12 point Times New Roman font) much like an annotated example in our text. State the problem and the desired solution. Detail each step in your solution, culminating in what you determined. Interpret your results.



L-01	L-02
□ square	square □
□ circular	circular □



L-01	L-02
□ square	square □
□ circular	circular □

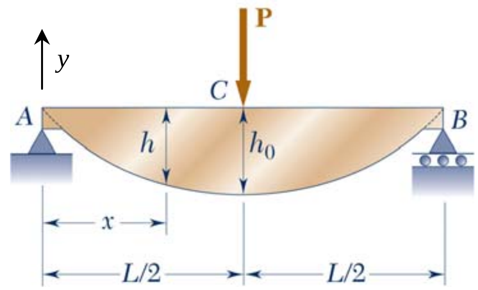


L-01	L-02
□ square	square □
□ circular	circular □

Images from Beer et al. Mechanics of materials.

EXAMPLE: Design a minimum weight simply supported beam under a midspan concentrated load P of square cross section of maximum dimension h_o and mass density ρ. Determine the performance ratio Π and compare that to the performance ratio Π_o of a prismatic "envelope" beam. (*ala* P5.128 BJDM 5e)

SOLUTION: The bending moment as a function of position along the beam is found by applying equilibrium to a free body diagram of a segment of the beam and is given by



$$M(x) = \frac{P}{2}x$$

over the left half of the beam $0 \le x \le L/2$. The maximum moment occurs under the concentrated load and is given by $M_{\text{max}} = M(x = L/2) = PL/4$. The maximum flexural stress σ_{max}, from the flexure formula, is set equal to an allowable stress σ_{allow} to determine the maximum load P_{max}

$$\sigma_{\text{max}} = \sigma_{\text{allow}} = \frac{M_{\text{max}} c}{I_z} = \frac{(P_{\text{max}} L/4)(h_o/2)}{h_o^4/12} \quad \Rightarrow \quad \underline{\underline{P_{\text{max}} = \frac{2h_o^3}{3L}\sigma_{\text{allow}}}}$$

where c is the maximum fiber distance measured from the neutral axis (*na*) and I_z is the moment of inertia with respect to the *na*. The volume of the envelope beam is $V_o = Lh_o^2$ such that its weight $W_o = \rho g V_o = \rho g L h_o^2$. The performance ratio is then

$$\Pi_o = \frac{P_{\text{max}}}{W_o} \quad \Rightarrow \quad \underline{\underline{\Pi_o = \frac{2h_o \sigma_{\text{allow}}}{3\rho g L^2}}}$$

A minimum weight beam will have equal maximum flexural stress in every cross section such that

$$\sigma_{\text{max}}(x) = \sigma_{\text{allow}} = \frac{M_{\text{max}} c(x)}{I(x)} = \frac{M_{\text{max}}}{S_z(x)} = \frac{3P_{\text{max}} x}{[h(x)]^3} \quad \Rightarrow \quad h(x) = \left(\frac{3P_{\text{max}}}{\sigma_{\text{allow}}}\right)^{1/3} x^{1/3}$$

where S_z is the section modulus and $h(x)$ is the entire height and width of each cross section.

The weight of the minimum weight beam is bounded as $L \to \infty$ by the weight of the envelope beam since

$$\lim_{x \to \infty} \frac{dh}{dx} = 0$$

Therefore, the ratio of the performance ratios $\Pi/\Pi_o \ge 1$.

A "soft" upper bound on the ratio of performance ratios can be found by considering two pyramids each of length $L/2$ and square cross section $h(x)$ by $h(x)$. The weight W_p of such a double pyramidal beam is

$$W_p = 2\rho g\left[\frac{1}{3}(h_o)(h_o)\left(\frac{L}{2}\right)\right] = \frac{1}{3}Lh_o^2$$

and the performance ratio of the double pyramidal beam is

$$\Pi \le \Pi_p = \frac{P_{\max}}{W_p} \quad \Rightarrow \quad \Pi \le \Pi_p = \frac{2h_o\sigma_{\text{allow}}}{\rho g L^2}$$

The ratio of the performance ratios is then

$$\frac{\Pi}{\Pi_o} \le \frac{\Pi_p}{\Pi_o} = 3$$

Therefore, the performance ratio for the minimum weight beam will be bounded such that $1 \le \Pi/\Pi_o \le 3$, and the specific value of Π will depend on the envelope of and allowable stress in the beam.

The following numerical values are used:

$$L = 1\text{ m} \qquad h_o = 25\text{ mm} \qquad \sigma_{\text{allow}} = 100\text{ MPa}$$

These values yield an allowable load of

$$P_{\max} = \frac{2h_o^3}{3L}\sigma_{\text{allow}} \quad \Rightarrow \quad P_{\max} = 1{,}042\text{ N}$$

and a performance ratio of $\Pi_o\rho g = 1.67\,\frac{\text{MN}}{\text{m}^3}$ for the envelope beam. The volume of the non-prismatic beam was determined using numerical integration and step size along the beam length of 10 mm, resulting in a performance ratio of $\Pi\rho g = 2.78\,\frac{\text{MN}}{\text{m}^3}$. The ratio of performance ratios is then

$$\boxed{\frac{\Pi}{\Pi_0} = 1.67}$$

The final step in this design would be to determine the cross sectional requirements near the supports where the internal shear becomes critical.